Discretizing the Advection of Differential Forms

Ralf Hiptmair, ETH

The behavior of electromagnetic fields in the stationary flow field of a conducting fluid flowing with velocity $\boldsymbol{v}$ can be modelled by the (non-dimensional) advection-diffusion equation

$$\underbrace{R_m^{-1}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{A}}_{\text{diffusion}} + \underbrace{\alpha\mathbf{A}}_{\text{dissipation}} + \underbrace{\mathbf{curl}\,\mathbf{A}\times\boldsymbol{v} + \mathbf{grad}\,(\mathbf{A}\cdot\boldsymbol{v})}_{\text{advection}} = \mathbf{j}_s \quad \text{in } \Omega\,. \tag{1}$$

Here, $\mathbf{A}$ stands a magnetic vector potential arising from temporal gauge, $\mathbf{j}_s$ is a source current and R_m is the so-called magnetic Reynolds number, which indicates the relative strength of magnetic diffusion compared to the advection with the fluid. For fast moving fluids, it can become very large, thus spawning advection dominated boundary value problems.

We observe that (1) is the vector proxy version of a member of a family of singularly perturbed evolution boundary value problems for differential ℓ-forms $\omega = \omega(t, \boldsymbol{x})$, $0 \leq \ell < d$:

$$\epsilon\, \mathsf{d} \star \mathsf{d}\, \omega + \alpha\omega + \star\mathcal{L}_{\boldsymbol{v}}\, \omega = \varphi \quad \text{in } \Omega \subset \mathbb{R}^d\,, \tag{2}$$

where d is the exterior derivative, $\mathcal{L}_{\boldsymbol{v}}$ the *Lie derivative* in the direction of $\boldsymbol{v}$, $\star$ designates a (Euclidean) Hodge operator, and ϵ can be very small. For $\ell = 1$ and $d = 3$, (2) agrees with (1), whereas for $\ell = 0$ we recover the well-known scalar convection-diffusion equation.

In light of (2), we aim for a discretization of (1) in the spirit of discrete exterior calculus (DEC), relying on discrete 1-forms for the approximation of $\mathbf{A}$, whose lowest order representatives are known as Whitney-1-forms (edge elements for $\ell = 1$). This offers a viable discretization of the diffusive terms in (1), but is no remedy for the notorious instabilities in convection-dominated situations marked by $\epsilon \approx 0$.

Inspired by successful schemes devised for the scalar case $\ell = 0$ we pursue a **stabilized Galerkin approach** in the spirit of discontinuous Galerkin methods with upwind numerical flux. We note that even if ω is approximated by means of discrete ℓ-forms, jump terms across interelement faces have to be retained for $\ell > 0$ and they hold the key to stability. Rigorous a priori convergence estimate are provided for the stationary problem in the limit case $\epsilon = 0$ and for Lipschitz continuous velocity fields.

For discontinuous velocities $\boldsymbol{v}$, existence of solutions of (2) is open for $\ell > 0$. However, an extension of a stabilized Galerkin scheme performs well in numerical experiments also in this case.

References

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Authors:

H. HEUMANN[1], R. HIPTMAIR[2], S. MISHRA [2], C. PAGLIANTINI[2]

[1] Laboratoire J.A.Dieudonné, UMR CNRS, Université de Nice Sophia-Antipolis
[2]Seminar for Applied Mathematics, ETH Zürich, Switzerland